CSE-217: Theory of Computation

Non Regular Language

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NONREGULAR LANGUAGE

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1. To understand the power of finite automata, one must also understand their limitations.

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2. Let's take the language $B = \{0^n 1^n | n \geq 0\}$.

NONREGULAR LANGUAGE

Example

Consider two languages over alphabet $\Sigma = \{0, 1\}$:

C = {w| w has an equal number of 0's and 1's}

D = {w| w has an equal number of occurrences of 01 and 10 as substrings}.

NONREGULAR LANGUAGE

D = {w| w has an equal number of occurrences of 01 and 10 as substrings}.

- ϵ

NONREGULAR LANGUAGE

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- ϵ belong
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- ϵ belong
- 0 belong
- 1

NONREGULAR LANGUAGE

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- ϵ belong
- 0 belong
- 1 belong
- 01

NONREGULAR LANGUAGE

D = {w| w has an equal number of occurrences of 01 and 10 as substrings}.

- ϵ belong
- 0 belong
- 1 belong
- 01 not belong
- 10

NONREGULAR LANGUAGE

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- ϵ belong
- 0 belong
- 1 belong
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- 10 not belong
- 010

NONREGULAR LANGUAGE

D = {w| w has an equal number of occurrences of 01 and 10 as substrings}.

- ϵ belong
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- 101

NONREGULAR LANGUAGE

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- ϵ belong
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- 1 belong
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- 10 not belong
- 010 belong
- 101 belong
- 0110

NONREGULAR LANGUAGE

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- ϵ belong
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- 0110 belong
- 01100 belong
- 1101110011

NONREGULAR LANGUAGE

D = {w| w has an equal number of occurrences of 01 and 10 as substrings}.

ϵ	belong
0	belong
1	belong
01	not belong
10	not belong
010	belong
101	belong
0110	belong
01100	belong
1101110011	belong

NONREGULAR LANGUAGE

D = {w| w has an equal number of occurrences of 01 and 10 as substrings}.

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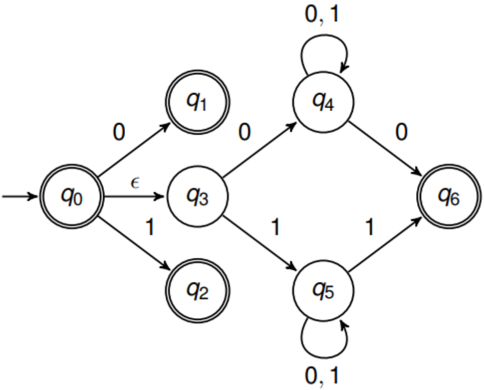

THE PUMPING LEMMA FOR REGULAR LANGUAGES

Pumping Lemma

1. The technique for proving nonregularity stems from a theorem called **the pumping lemma.**
2. This theorem states that all regular languages have a special property.
3. If a language does not have this property, we are guaranteed that it is not regular.

The Pumping Lemma for Regular Languages

Sipser, 1.4, p-77

THEOREM 1.70

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

- $|s|$ represents the length of string s.
- y^i means that i copies of y are concatenated together.
- y^0 equals ϵ.

The Pumping Lemma for Regular Languages

Sipser, 1.4, p-77

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1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

- When *s* is divided into *xyz*, either *x* or *z* may be ϵ.
- But condition 2 says that $y \neq \epsilon$.
- Without condition 2 the theorem would be trivially true.

The Pumping Lemma for Regular Languages

Sipser, 1.4, p-77

THEOREM 1.70

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

- Condition 3 states that the pieces x and y together have length at most p.
- It is an extra technical condition that we occasionally find useful when proving certain languages to be nonregular.

The Pumping Lemma for Regular Languages — *continued*

Sipser, 1.4, p-77

PROOF IDEA

- Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA that recognizes A.
- We assign the pumping length p to be the number of states of M.
- We show that any string s in A of length at least p may be broken into the three pieces xyz, satisfying our three conditions.
- What if no strings in A are of length at least p?
- Then our task is even easier because the theorem becomes vacuously true.
- Obviously the three conditions hold for all strings of length at least p if there aren't any such strings.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

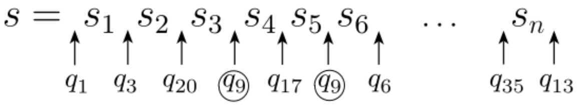


FIGURE 1.71
Example showing state q_9 repeating when M reads s

- If s in A has length at least p, consider the sequence of states that M goes through when computing with input s.
- It starts with q_1 the start state, then goes to, say, q_3, then, say, q_{20}, then q_9, and so on, until it reaches the end of s in state q_{13}.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

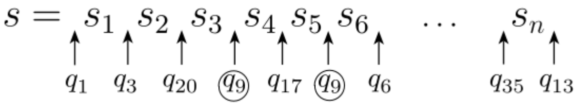


FIGURE 1.71
Example showing state q_9 repeating when M reads s

- With s in A, we know that M accepts s, so q_{13} is an accept state.
- If we let n be the length of s, the sequence of states $q_1, q_3, q_{20}, q_9, \ldots, q_{13}$ has length $n + 1$.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

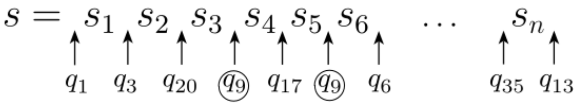


FIGURE 1.71
Example showing state q_9 repeating when M reads s

- Because *n* is at least *p*, we know that $n+1$ is greater than *p*, the number of states of *M*.
- Therefore, the sequence must contain a repeated state.
- This result is an example of the pigeonhole principle.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

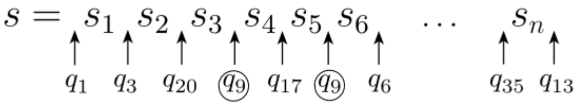


FIGURE 1.71

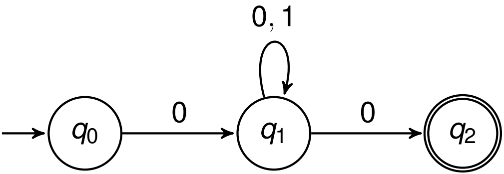
Example showing state q_9 repeating when M reads s

- State q_9 is the one that repeats.

The Pumping Lemma for Regular Languages

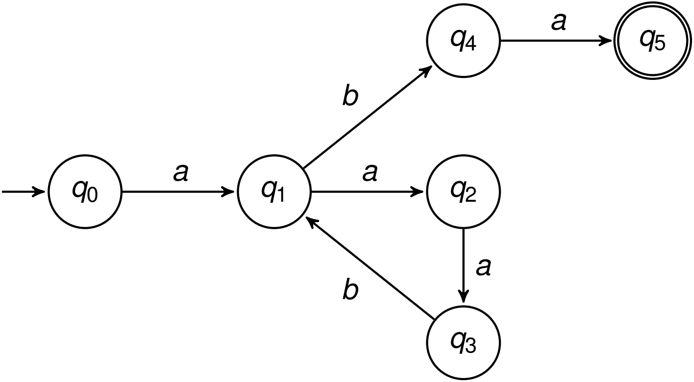
$L = \{w \mid w \text{ starts and ends with } 0, |w| \geq 2\}$

$L = 0\Sigma^*0$



The Pumping Lemma for Regular Languages

$L = a(aab)^*ba$



The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

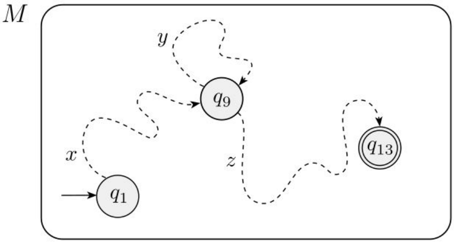


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- Piece x is the part of s appearing before q_9.
- Piece y is the part between the two appearances of q_9.
- Piece z is the remaining part of s, coming after the second occurrence of q_9.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

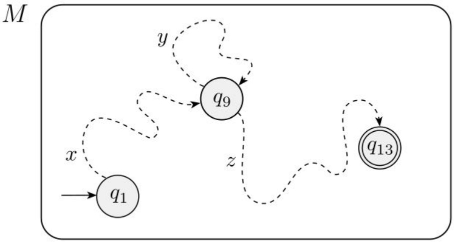


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- x takes M from the state q_1 to q_9.
- y takes M from q_9 back to q_9.
- z takes M from q_9 to the accept state q_{13}.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

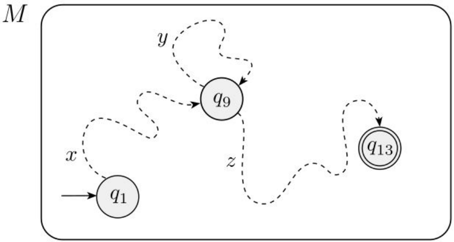


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- Suppose that we run M on input $xyyz$.
- We know that x takes M from q_1 to q_9.

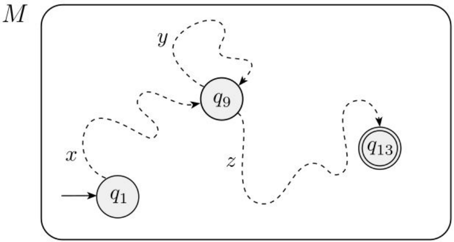


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- Then the first y takes it from q_9 back to q_9 , as does the second y.
- Then z takes it to q_{13}.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

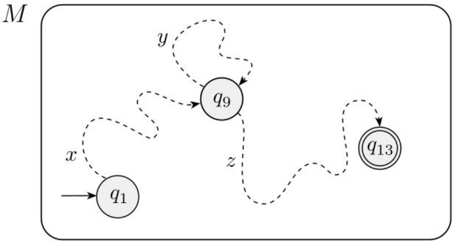


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- With q_{13} being an accept state, M accepts input $xyyz$.
- Similarly, it will accept xy^iz for any $i > 0$.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

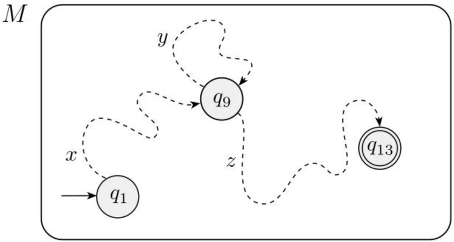


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- For the case $i = 0$, $xy^i z = xz$, which is accepted for similar reasons.
- That establishes condition 1.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

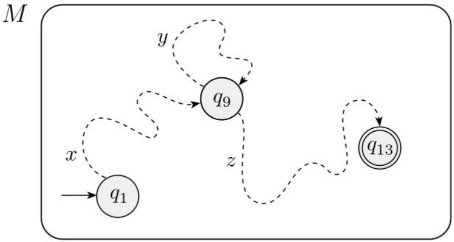


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- Checking condition 2, we see that $|y| > 0$, as it was the part of s that occurred between two different occurrences of state q_9.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

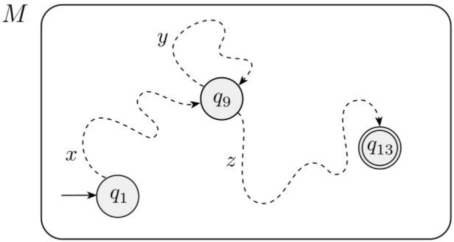


FIGURE 1.72
Example showing how the strings x, y, and z affect M

- In order to get condition 3, we make sure that q_9 is the first repetition in the sequence.
- By the pigeonhole principle, the first $p + 1$ states in the sequence must contain a repetition.
- Therefore, $|xy| \leq p$.

PROOF

- Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA recognizing A and p be the number of states of M.
- Let $s = s_1 s_2 \dots s_n$ be a string in A of length n, where $n \geq p$.
- Let $r_1, r_2, \dots, r_{n+1}$ be the sequence of states that M enters while processing s.
- So $r_{i+1} = \delta(r_i, s_i)$ for $1 \geq i \geq n$.
- This sequence has length $n + 1$, which is at least $p + 1$.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-77

- Among the first $p+1$ elements in the sequence, two must be the same state.
- By the pigeonhole principle. We call the first of these r_j and the second r_ℓ.
- Because r_ℓ occurs among the first $p+1$ places in a sequence starting at r_1, we have $\ell \geq p+1$.

The Pumping Lemma... — *continued*

Sipser, 1.4, p-77

- Let $x = s_1 \ldots s_{j-1}$.
- $y = s_j \ldots s_{\ell-1}$.
- $z = s_\ell \ldots s_n$.

The Pumping Lemma... — *continued*

Sipser, 1.4, p-77

- Let $x = s_1 \dots s_{j-1}$.
- $y = s_j \dots s_{\ell-1}$.
- $z = s_\ell \dots s_n$.
- x takes M from r_1 to r_j.
- y takes M from r_j to r_j.
- z takes M from r_j to r_{n+1}, which is an accept state, M must accept xy^iz for $i \geq 0$.

The Pumping Lemma… — *continued*

Sipser, 1.4, p-77

- We know that $j \neq \ell$, so $|y| > 0$.
- $\ell \geq p + 1$, so $|y| > 0$.
- $\ell \geq p + 1$, so $|xy| \geq p$.
- Thus we have satisfied all conditions of the pumping lemma.

The Pumping Lemma… — *continued*

Sipser, 1.4, p-80

- To use the pumping lemma to prove that a language B is not regular, first assume that B is regular in order to obtain a contradiction.
- Then use the pumping lemma to guarantee the existence of a pumping length p such that all strings of length p or greater in B can be pumped.

- Next, find a string s in B that has length p or greater but that cannot be pumped.

- Finally, demonstrate that s cannot be pumped by considering all ways of dividing s into x, y, and z (taking condition 3 of the pumping lemma into account if convenient).
- For each such division, find a value i where $xy^i z \notin B$.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-80

- This final step often involves grouping the various ways of dividing s into several cases and analyzing them individually.
- The existence of s contradicts the pumping lemma if B were regular.
- Hence B cannot be regular.

The Pumping Lemma. . . — *continued*

Sipser, 1.4, p-80

- Finding s sometimes takes a bit of creative thinking.
- You may need to hunt through several candidates for s before you discover one that works.
- Try members of B that seem to exhibit the “essence” of B’s nonregularity.

Example

Sipser, Example 1.73, p-80

- Let B be the language $\{0^n 1^n \mid n \geq 0\}$.
- We use the pumping lemma to prove that B is not regular.
- The proof is by contradiction.

Example — *continued*

Sipser, Example 1.73, p-80

- Assume to the contrary that B is regular.
- Let p be the pumping length given by the pumping lemma.

Example — *continued*

Sipser, Example 1.73, p-80

- Choose s to be the string 0^p1^p.
- Because s is a member of B and s has length more than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$.
- Where for any $i \geq 0$ the string xy^iz is in B.

Example — *continued*

Sipser, Example 1.73, p-80

- We consider three cases to show that this result is impossible.

---

1. The string y consists only of 0s.

- In this case, the string $xyyz$ has more 0s than 1s and so is not a member of B, violating condition 1 of the pumping lemma.
- This case is a contradiction.

Example — *continued*

Sipser, Example 1.73, p-80

- We consider three cases to show that this result is impossible.

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2. The string y consists only of 1s.

- This case also gives a contradiction.

Example — *continued*

Sipser, Example 1.73, p-80

- We consider three cases to show that this result is impossible.

---

3. The string y consists of both 0s and 1s.

- In this case, the string $xyyz$ may have the same number of 0s and 1s, but they will be out of order with some 1s before 0s.
- Hence it is not a member of B, which is a contradiction.

Example — *continued*

Sipser, Example 1.73, p-80

- Thus a contradiction is unavoidable if we make the assumption that B is regular.
- So B is not regular.

Example

Sipser, Example 1.74, p-80

- $C = \{w \mid w \text{ has an equal number of 0s and 1s}\}$.
- We use the pumping lemma to prove that C is not regular.
- The proof is by contradiction.

Example — *continued*

Sipser, Example 1.74, p-80

- Assume to the contrary that C is regular.
- Let p be the pumping length given by the pumping lemma.
- Let s be the string 0^p1^p.
- With s being a member of C and having length more than p, the pumping lemma guarantees that s can be split into three pieces.
- $s = xyz$, where for any $i \geq 0$ the string xy^iz is in C.

Example — *continued*

Sipser, Example 1.74, p-80

- We would like to show that this outcome is impossible.

Example — *continued*

Sipser, Example 1.74, p-80

- But wait, it is possible!
- If we let x and z be the empty string and y be the string 0^p1^p, then xy^iz always has an equal number of 0s and 1s and hence is in C.
- So it seems that s can be pumped.

Example — *continued*

Sipser, Example 1.74, p-80

- Here condition 3 in the pumping lemma is useful.
- It stipulates that when pumping s, it must be divided so that $|xy| \leq p$.
- That restriction on the way that s may be divided makes it easier to show that the string $s = 0^p 1^p$ we selected cannot be pumped.
- If $|xy| \leq p$, then y must consist only of 0s, so $xyyz \notin C$.
- Therefore, s cannot be pumped.
- That gives us the desired contradiction.

Example

Sipser, Example 1.75, p-81

- $F = \{ww \mid w \in \{0,1\}^*\}$.
- We use the pumping lemma to prove that F is not regular.

Example — *continued*

Sipser, Example 1.75, p-81

- Assume to the contrary that F is regular.
- Let p be the pumping length given by the pumping lemma.
- Let s be the string $0^p 10p^1$.
- Because s is a member of F and s has length more than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, satisfying the three conditions of the lemma.

Example — *continued*

Sipser, Example 1.75, p-81

- We show that this outcome is impossible.
- Condition 3 is once again crucial because without it we could pump s if we let x and z be the empty string.
- With condition 3 the proof follows because y must consist only of 0s, so $xyyz \notin F$.

Example — *continued*

Sipser, Example 1.75, p-81

- Observe that we chose $s = 0^p10^p1$ to be a string that exhibits the “essence” of the nonregularity of F, as opposed to, say, the string 0^p0^p.
- Even though 0^p0^p is a member of F, it fails to demonstrate a contradiction because it can be pumped.

Example

Sipser, Example 1.76, p-82

- We demonstrate a nonregular unary language.
- $D = \left\{ 1^{n^2} \mid n \geq 0 \right\}$.
- We use the pumping lemma to prove that D is not regular.
- The proof is by contradiction.

Example — *continued*

Sipser, Example 1.76, p-82

- Assume to the contrary that D is regular.
- Let p be the pumping length given by the pumping lemma.

Example — *continued*

Sipser, Example 1.76, p-82

- Let s be the string 1^{p^2}.
- Because s is a member of D and s has length at least p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$.
- where for any $i \geq 0$ the string xy^iz is in D.

Example — *continued*

Sipser, Example 1.76, p-82

- We show that this outcome is impossible.
- The sequence of perfect squares:

$$0, 1, 4, 9, 16, 25, 36, 49, \ldots$$

- Note the growing gap between successive members of this sequence.
- Large members of this sequence cannot be near each other.

Example — *continued*

Sipser, Example 1.76, p-82

- Now consider the two strings xyz and xy^2z.
- These strings differ from each other by a single repetition of y.
- Consequently their lengths differ by the length of y.
- By condition 3 of the pumping lemma, $|xy| \leq p$ and thus $|y| \leq p$.

Example — *continued*

Sipser, Example 1.76, p-82

- We have $|xyz| = p^2$ and so $|xy^2z| \leq p^2 + p$.
- But $p^2 + p < p^2 + 2p + 1 = (p + 1)2$.
- Moreover, condition 2 implies that y is not the empty string and so $|xy^2z| > p^2$.
- Therefore, the length of xy^2z lies strictly between the consecutive perfect squares p^2 and $(p + 1)2$.
- Hence this length cannot be a perfect square itself.
- So we arrive at the contradiction $xy^2z \notin D$ and conclude that D is not regular.

Example

Sipser, Example 1.77, p-82

- Let E be the language $\{0^i1^j \mid i > j\}$.
- We use the pumping lemma to prove that E is not regular.
- The proof is by contradiction.

Example — *continued*

Sipser, Example 1.77, p-82

- Assume that E is regular.
- Let p be the pumping length for E given by the pumping lemma.

Example — *continued*

Sipser, Example 1.77, p-82

- Let $s = 0^{p+1}1^p$.
- Then s can be split into xyz, satisfying the conditions of the pumping lemma.
- By condition 3, y consists only of 0s.

Example — *continued*

Sipser, Example 1.77, p-82

- Let's examine the string $xyyz$ to see whether it can be in E.
- Adding an extra copy of y increases the number of 0s.
- But, E contains all strings in 01 that have more 0s than 1s.
- So increasing the number of 0s will still give a string in E.
- No contradiction occurs.

Example — *continued*

Sipser, Example 1.77, p-82

- We need to try something else.
- The pumping lemma states that $xy^iz \in E$ even when $i = 0$.

Example — *continued*

Sipser, Example 1.77, p-82

- So let's consider the string $xy^0z = xz$.
- Removing string y decreases the number of 0s in s.
- Recall that s has just one more 0 than 1.
- Therefore, xz cannot have more 0s than 1s, so it cannot be a member of E.
- Thus we obtain a contradiction.

Example

Hopcroft, Motwani, and Ullman, Example 4.3, p-129

- Let us show that the language L_{pr} consisting of all strings of 1's whose length is a prime is not a regular language.
- Suppose it were.
- Then there would be a constant n satisfying the conditions of the pumping Lemma.

Example — *continued*

Hopcroft, Motwani, and Ullman, Example 4.3, p-129

- Consider some prime $p \geq n + 2$.
- There must be such a p, since there are an infinity of primes.

Example — *continued*

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- Let $w = 1^r$.
- By the pumping lemma, we can break $w = xyz$ such that $y \neq \epsilon$ and $|xy| \leq n$.
- Let $|y| = m$.
- Then $|xz| = p - m$.

Example — *continued*

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- Now consider the string $xy^{p-m}z$.
- This must be in L_{pr} by the pumping lemma, if L_{pr} really is regular.
- However,

$$\begin{aligned} |xy^{p-m}z| &= |xz| + (p-m)|y| \\ &= p - m + (p-m)m \\ &= (m+1)(p-m) \end{aligned}$$

- It looks like $|xy^{p-m}z|$ is not a prime, since it has two factors $(m+1)$ and $(p-m)$.

Example — *continued*

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- However, we must check that neither of these factors are 1.
- Since then $(m+1)(p-m)$ might be a prime after all.
- But $m+1 > 1$, since $y \neq \epsilon$ tells us $m \geq 1$.
- Also, $p-m \geq 1$, since $p \geq n+2$ was chosen, and $m \leq n$ since

$$m = |y| \leq |xy| \leq n$$

- Thus, $p-m \geq 2$.

Example — *continued*

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- Again we have started by assuming the language in question was regular.
- We derived a contradiction by showing that some string not in the language was required by the pumping lemma to be in the language.
- Thus, we conclude that L_{pr} is not a regular language.


It's over now!