# ICS141: Discrete Mathematics for Computer Science I

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# Lecture 20

## Chapter 4. Induction and Recursion

4.1 Mathematical Induction

4.2 Strong Induction

# Mathematical Induction



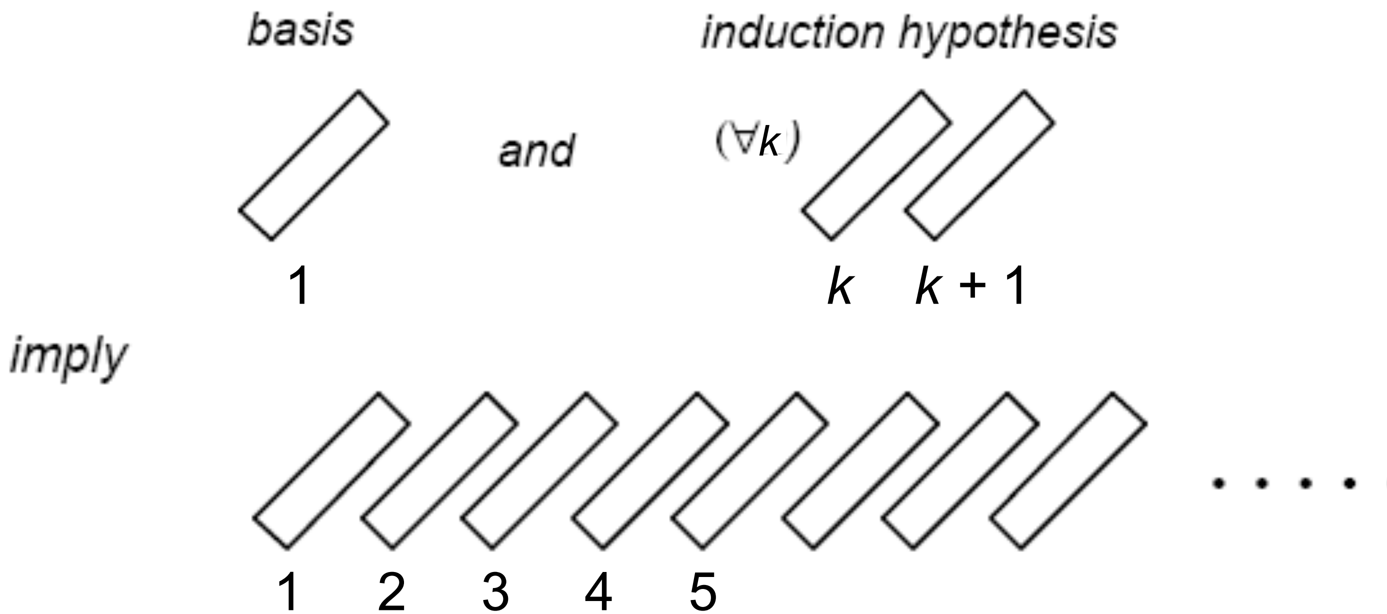
- A powerful, rigorous technique for proving that a statement $P(n)$ is true for ***every*** positive integers $n$, no matter how large.
- Essentially a "domino effect" principle.
- Based on a predicate-logic inference rule:

$$\frac{\begin{array}{l} P(1) \\ \forall k \geq 1\ [P(k) \rightarrow P(k+1)] \end{array}}{\therefore \forall n \geq 1\ P(n)}$$

***"The First Principle of Mathematical Induction"***

# The "Domino Effect"

- **Premise #1:** Domino #1 falls.
- **Premise #2:** For every $k \in \mathbf{Z}^+$, if domino #$k$ falls, then so does domino #$k$+1.
- **Conclusion:** All of the dominoes fall down!




**Note:** this works even if there are infinitely many dominoes!

# Mathematical Induction Recap.

- **PRINCIPLE OF MATHEMATICAL INDUCTION:**

  To prove that a statement $P(n)$ is true for all positive integers $n$, we complete two steps:

  - **BASIS STEP**: Verify that $P(1)$ is true
  - **INDUCTIVE STEP**: Show that the conditional statement $\underbrace{P(k)}_{\text{Inductive Hypothesis}} \rightarrow P(k+1)$ is true for all positive integers $k$

# Validity of Induction

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**Proof:** that $\forall n \geq 1\ P(n)$ is a valid consequent:

Given any $k \geq 1$, the 2nd premise

$\forall k \geq 1\ (P(k) \rightarrow P(k+1))$ trivially implies that

$(P(1) \rightarrow P(2)) \land (P(2) \rightarrow P(3)) \land \ldots \land (P(n-1) \rightarrow P(n))$.

Repeatedly applying the hypothetical syllogism rule to adjacent implications in this list $n - 1$ times then gives us $P(1) \rightarrow P(n)$; which together with $P(1)$ (premise #1) and *modus ponens* gives us $P(n)$.

Thus $\forall n \geq 1\ P(n)$. ■

# Outline of an Inductive Proof

- Let us say we want to prove $\forall n \in \mathbf{Z}^+ \ P(n)$.
  - Do the ***base case*** (or ***basis step***):
    Prove $P(1)$.
  - Do the ***inductive step***:
    Prove $\forall k \in \mathbf{Z}^+ \ P(k) \rightarrow P(k+1)$.
    - *E.g.* you could use a direct proof, as follows:
    - Let $k \in \mathbf{Z}^+$, assume $P(k)$. (*inductive hypothesis*)
    - Now, under this assumption, prove $P(k+1)$.
  - The inductive inference rule then gives us
    $\forall n \in \mathbf{Z}^+ \ P(n)$.

# Induction Example

- Show that, for $n \geq 1$

$$\boxed{1+2+\cdots+n=\frac{n(n+1)}{2}}$$

- Proof by induction
  - $P(n)$: the sum of the first $n$ positive integers is $n(n+1)/2$, i.e. $P(n)$ is
  - ***Basis step***: Let $n = 1$. The sum of the first positive integer is 1, i.e. $P(1)$ is true.

$$1=\frac{1(1+1)}{2}$$

# Example *(cont.)*

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- ***Inductive step***: Prove $\forall k \geq 1: P(k) \rightarrow P(k+1)$.
  - Inductive Hypothesis, $P(k)$:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

  - Let $k \geq 1$, assume $P(k)$, and prove $P(k+1)$, i.e.

$$\underbrace{1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}}_{P(k+1)}$$

**This is what you have to prove**

# Example (cont.)

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- ***Inductive step*** continues…

By inductive hypothesis $P(k)$

$$
\underbrace{1+2+\cdots+k+(k+1)}_{P(k+1)} = \frac{k(k+1)}{2}+(k+1)
$$

$$
= \frac{k(k+1)}{2}+\frac{2(k+1)}{2}
$$

$$
= \frac{k^2+3k+2}{2}
$$

$$
= \frac{(k+1)(k+2)}{2}
$$

- Therefore, by the principle of mathematical induction $P(n)$ is true for all integers $n$ with $n \geq 1$

# Induction Example 2

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- **Example 2**: Conjecture a formula for the sum of the first *n* positive odd integers. Then prove your conjecture using mathematical induction.
  - Practical Method for General Problem Solving.

    Special Case: Deriving a Formula

    Step 1. Calculate the result for some small cases

    Step 2. Guess a formula to match all those cases

    Step 3. Verify your guess in the general case

# Example 2 (cont.)

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- Step 1: Examine small cases

$$1 = 1 = 1^2$$
$$1 + 3 = 4 = 2^2$$
$$1 + 3 + 5 = 9 = 3^2$$
$$1 + 3 + 5 + 7 = 16 = 4^2$$

- Step 2: It sure looks like $1 + 3 + \cdots + (2n-1) = n^2$
- Step 3: Try to prove this assertion by induction

$$\boxed{\forall n \geq 1,\ \sum_{i=1}^{n} (2i-1) = n^2}$$

# Example *Continues*…

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- Prove that the sum of the first *n* odd positive integers is $n^2$. That is, prove:

$$\forall n \geq 1, \ \underbrace{\sum_{i=1}^{n} (2i-1) = n^2}_{P(n)}$$

- Proof by induction.
  - ***Basis step***: Let $n = 1$. The sum of the first 1 odd positive integer is 1 which equals $1^2$. i.e. $P(1)$ is true.

# Example *Continues*…

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- ***Inductive step***: Prove $\forall k \geq 1: P(k) \rightarrow P(k+1)$.
  - Inductive Hypothesis, $P(k)$: $\left[\sum_{i=1}^{k}(2i-1)=k^2\right]$
  - Let $k \geq 1$, assume $P(k)$, and prove $P(k+1)$.

$$\sum_{i=1}^{k+1}(2i-1)=\left(\sum_{i=1}^{k}(2i-1)\right)+\left(2(k+1)-1\right)$$

$$=k^2+\left(2k+1\right)$$

***By inductive hypothesis P(k)***

$$=(k+1)^2$$

# Induction Example 3

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- Prove that $\forall n \geq 1,\ n < 2^n$. Let $P(n) = (n < 2^n)$
  - ***Basis step***: $P(1)$: $(1 < 2^1) \equiv (1 < 2)$: **True**.
  - ***Inductive step***: For $k \geq 1$, prove $P(k) \rightarrow P(k+1)$.
    - Assuming $k < 2^k$, prove $k + 1 < 2^{k+1}$.
    - Note $k + 1 < 2^k + 1$ (by inductive hypothesis)
      $< 2^k + 2^k$ (because $1 < 2^k$ for $k \geq 1$)
      $= 2 \cdot 2^k = 2^{k+1}$
    - So $k + 1 < 2^{k+1}$, i.e. $P(k+1)$ is true
  - Therefore, by the principle of mathematical induction $P(n)$ is true for all integers $n$ with $n \geq 1$.

# Generalizing Induction



- Rule can also be used to prove $\forall n \geq c\ P(n)$ for a given constant $c \in \mathbf{Z}$, where maybe $c \neq 1$.
  - In this circumstance,

    the basis step is to prove $P(c)$ rather than $P(1)$,

    and the inductive step is to prove

    $\forall k \geq c\ (P(k) \rightarrow P(k+1))$.

# Induction Example 4

- **Example 6**: Prove that $2^n < n!$ for $n \geq 4$ using mathematical induction.
  - $P(n)$: $2^n < n!$
  - ***Basis step***: Show that $P(4)$ is true
    - Since $2^4 = 16 < 4! = 24$, $P(4)$ is true
  - ***Inductive step***: Show that $P(k) \rightarrow P(k+1)$ for $k \geq 4$
    - $2^{k+1} = 2 \cdot 2^k$ (by definition of exponent)
      $< 2 \cdot k!$ (by the inductive hypothesis $P(k)$)
      $< (k + 1) \cdot k!$ (because $2 < k+1$ for $k \geq 4$ )
      $= (k + 1)!$ (by definition of factorial function)

    $P(k+1)$ is true
  - Therefore, by the principle of mathematical induction $P(n)$ is true for all integers $n$ with $n \geq 4$.

# Second Principle of Induction

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a.k.a. "Strong Induction"

- Characterized by another inference rule:

$$\begin{array}{l} P(1) \\ \forall k \geq 1\text{: } (\overbrace{P(1) \wedge P(2) \wedge \cdots \wedge P(k)}^{P \text{ is true in } all \text{ previous cases}}) \rightarrow P(k+1) \\ \hline \therefore \forall n \geq 1\text{: } P(n) \end{array}$$

- The only difference between this and the 1st principle is that:
  - the inductive step here makes use of the stronger hypothesis that all of $P(1)$, $P(2)$,…, $P(k)$ are true, not just $P(k)$.

# Example of Second Principle

- Show that every integer $n > 1$ can be written as a product $n = p_1 p_2 \ldots p_s = \prod p_i$ of some series of $s$ prime numbers.
  - Let $P(n)$ = "$n$ has that property "
- ***Basis step***: $n = 2$, let $s = 1$, $p_1 = 2$. Then $n = p_1$
- ***Inductive step***: Let $k \geq 2$. Assume $\forall 2 \leq i \leq k: P(i)$.
  - Consider $k + 1$. If it's prime, let $s = 1$, $p_1 = k + 1$.
  - Else $k + 1 = ab$, where $1 < a \leq k$ and $1 < b \leq k$. Then $a = p_1 p_2 \ldots p_t$ and $b = q_1 q_2 \ldots q_u$.
    (by Inductive Hypothesis)
    Then we have that $k + 1 = p_1 p_2 \ldots p_t q_1 q_2 \ldots q_u$, a product of $s = t + u$ primes.

# Generalizing Strong Induction

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- Handle cases where the inductive step is valid only for integers greater than a particular integer
  - $P(n)$ is true for $\forall n \geq b$ ($b$: fixed integer)

- **BASIS STEP**: Verify that $P(b), P(b+1), \ldots, P(b+j)$ are true ($j$: a fixed positive integer)

- **INDUCTIVE STEP**: Show that the conditional statement $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers $k \geq b + j$

# Another 2nd Principle Example

- Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
  - $P(n)$ = "postage of $n$ cents can be formed using 4-cent and 5-cent stamps" for $n \geq 12$.
  - ***Basis step***:
    - $12 = 3 \cdot 4$
    - $13 = 2 \cdot 4 + 1 \cdot 5$
    - $14 = 1 \cdot 4 + 2 \cdot 5$
    - $15 = 3 \cdot 5$
    - So $\forall 12 \leq i \leq 15, P(i)$.

# Example *(cont.)*

- ***Inductive step***:
  - Let $k \geq 15$, assume $\forall 12 \leq i \leq k, P(i)$.
  - Note $12 \leq k - 3 \leq k$, so $P(k - 3)$.

    (by inductive hypothesis)

    This means we can form postage of $k$ – 3 cents using just 4-cent and 5-cent stamps.
  - Add a 4-cent stamp to get postage for $k + 1$, i.e. $P(k + 1)$ is true (postage of $k$ + 1 cents can be formed using 4-cent and 5-cent stamps).
- Therefore, by the 2nd principle of mathematical induction $P(n)$ is true for all integers $n$ with $n \geq 12$.

# Another 2nd Principle Example

- Prove by the 1st Principle.
  - $P(n)$ = "postage of $n$ cents can be formed using 4-cent and 5-cent stamps", $n \geq 12$.
  - ***Basis step***: $P(12)$: $12 = 3 \cdot 4$.
  - ***Inductive step***: $P(k) \rightarrow P(k+1)$
    - Case 1: At least one 4-cent stamp was used for $P(k)$
      - $k + 1 = k - 4 + 5$ (i.e. replace the 4-cent stamp with a 5-cent stamp to form a postage of $k + 1$ cents)

# Example *Continues*…

- ***Inductive step***: $P(k) \rightarrow P(k+1)$
  - Case 2: No 4-cent stamps were used for $P(k)$
    - Since $k \geq 12$, at least three 5-cent stamps are needed to form postage of $k$ cents
    - $k + 1 = k - 3 \cdot 5 + 4 \cdot 4$ (i.e. replace three 5-cent stamps with four 4-cent stamps to form a postage of $k + 1$ cents)

- Therefore, by the principle of mathematical induction $P(n)$ is true for all integers $n$ with $n \geq 12$.

# The Well-Ordering Property

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- Another way to prove the validity of the inductive inference rule is by using the *well-ordering property*, which says that:
  - Every non-empty set of non-negative integers has a minimum (smallest) element.
  - $\forall \varnothing \subset S \subseteq \mathbf{N}$: $\exists m \in S$ such that $\forall n \in S$, $m \leq n$
- This implies that $\{n | \neg P(n)\}$ (if non-empty) has a minimum element $m$, but then the assumption that $P(m-1) \rightarrow P((m-1)+1)$ would be contradicted.